Fixed Point Results for Monotone Nonexpansive Mappings in MS

Buthinah A. Bin Dehaish
King Abdulaziz University, Jeddah, Saudi Arabia
E-mail: bbendehaish@kau.edu.sa

Mathematical Optimisation Down Under (MODU2016) workshop
RMIT University
Melbourne - Australia,
July 18-22, 2016
Let $X$ be a metric space and $T : C \to C$ be a monotone nonexpansive mapping on a nonempty, bounded, closed, and convex subset of $X$. In this talk, we will show that if $X$ is a uniformly convex metric space, then $T$ has a fixed point.
In 1922 Banach published his fixed point theorem also known as **Banach’s Contraction Principle** which uses the concept of Lipschitz mappings.

**Definition**

Let $(M, d)$ be a metric space. The map $T : M \to M$ is said to be **Lipschitzian** if there exists a constant $k > 0$ (called Lipschitz constant) such that

$$d\left( T(x), T(y) \right) \leq k \, d(x, y)$$

for all $x, y \in M$. A Lipschitzian mapping with a Lipschitz constant $k$ less than 1, i.e. $k < 1$, is called **contraction**, and **nonexpansive** when $k = 1$. 
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**Theorem**

Let \((M, d)\) be a complete metric space and let \(T : M \to M\) be a contraction mapping. Then \(T\) has a unique fixed point \(\omega\), and for each \(x \in M\), we have

\[
\lim_{n \to \infty} T^n(x) = \omega
\]

Moreover, for each \(x \in M\), we have

\[
d\left(T^n(x), \omega\right) \leq \frac{k^n}{1 - k} d\left(T(x), x\right).
\]
Nonexpansive mappings are those mappings which have Lipschitz constant equal to one. Their investigation remain a popular area of research in various fields.

\[^{1}\text{F. E. Browder,}\ Nonexpansive\ nonlinear\ operators\ in\ a\ Banach\ space,\ Proc.\ Nat.\ Acad.\ Sci.\ U.S.A.,\ 54\ (1965),\ 1041-1044.\]
\[^{2}\text{D. Göhde,}\ Zum\ Prinzip\ der\ kontraktiven\ Abbildung,\ Math.\ Nachr.\ 30\ (1965),\ 251-258.}\]
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\textbf{Theorem}

\textit{If }$K$\textit{ is a bounded closed convex subset of a uniformly convex Banach space }$X$\textit{ and if }$T : K \to K$\textit{ is nonexpansive, then }$T$\textit{ has a fixed point. Moreover the fixed point set of }$T$\textit{ is a closed convex subset of }$K$\textit{.}


The existence part of this result was also obtained by Kirk\textsuperscript{1}: under slightly weaker assumptions.

Since then several fixed point theorems for nonexpansive mappings in Banach spaces have been derived.

\textsuperscript{1}W. A. Kirk, \textit{A fixed point theorem for mappings which do not increase distances}, Amer. Math. Monthly 72(1965), 1004-1006.
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It took few decades to extend the fixed point theory of nonexpansive mappings to nonlinear domains such as metric spaces.

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Recently a new direction has been developed when the Lipschitz condition is satisfied only for comparable elements in a partially ordered metric space. This approach was initiated by Ran and Reurings\textsuperscript{1}. In particular, they showed how this extension is useful when dealing with some special matrix equations.

\textsuperscript{1}Ran and Reurings, \textit{A fixed point theorem in partially ordered sets and some applications to matrix equations}, Proc. of the AMS (2004), 14351443.

Recently a new direction has been developed when the Lipschitz condition is satisfied only for comparable elements in a partially ordered metric space. This approach was initiated by Ran and Reurings\(^1\). In particular, they showed how this extension is useful when dealing with some special matrix equations.

The study those equations is motivated by the fact that they often arise in the analysis of ladder networks, dynamic programming, control theory and many other applications\(^2\).

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\(^1\) Ran and Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. of the AMS (2004), 1435-1443.  
Another application of Ran and Reurings fixed point result was used by Nieto and Rodríguez-López \(^1\) to find a periodic solution to a differential equation.

\(^1\)Nieto and Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order 22: 3 (2005), 223-239.
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Similarly and following the extension of the Banach Contraction Principle to the case of metric spaces endowed with a partial order, it was natural to try to investigate the case of nonexpansive mappings into such metric spaces.

\(^1\)Nieto and Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order 22: 3 (2005), 223-239.
The aim of this work is to investigate the existence of fixed points of monotone nonexpansive mappings. In particular, we prove that if $X$ is a uniformly convex hyperbolic metric space, then any monotone nonexpansive mapping defined on a nonempty bounded convex subset has a fixed point.
Most of the classical metric spaces are also partially ordered. The distance and the partial order do share some properties which happen to be very useful.

Let \((M, d)\) be a metric space endowed with a partial order \(\preceq\). We will say that \(x, y \in M\) are **comparable** whenever \(x \preceq y\) or \(y \preceq x\).
Basic definitions

Definition

Let \((M, d, \preceq)\) be a Banach space endowed with a partial order. Let \(T : M \to M\) be a map. \(T\) is said to be monotone or order-preserving if

\[ x \preceq y \implies T(x) \preceq T(y), \]

for any \(x, y \in M\).
Definition

Let \((M, d, \preceq)\) be a Banach space endowed with a partial order. Let \(T : M \to M\) be a map. \(T\) is said to be **monotone Lipschitzian** mapping if \(T\) is monotone and there exists \(k \geq 0\) such that

\[
d(T(x), T(y)) \leq k \, d(x, y),
\]

for any \(x, y \in M\) such that \(x\) and \(y\) are comparable. If \(k < 1\), then we say that \(T\) is a **monotone contraction** mapping. And if \(k = 1\), \(T\) is called a **monotone nonexpansive** mapping.

A point \(x \in C\) is said to be a **fixed point of** \(T\) if \(T(x) = x\). The set of fixed points of \(T\) is denoted by \(\text{Fix}(T)\).

Note that monotone Lipschitzian mappings are not necessarily continuous.
Recall that an order interval is any of the subsets

\[ [a, \to) = \{ x \in X; a \preceq x \} \quad \text{or} \quad (\leftarrow, a] = \{ x \in X; x \preceq a \}, \]

for any \( a \in X \). The closed order interval \([a, b]\) is defined by

\[ [a, b] = \{ x \in X; a \preceq x \preceq b \} = [a, \to) \cap (\leftarrow, b] \]
In this section, we will establish Browder and Gähde’s fixed point theorem for monotone nonexpansive mappings. The setting will be uniformly convex hyperbolic metric spaces.
Let \((X, d)\) be a metric space. Suppose that there exists a family \(F\) of metric segments such that any two points \(x, y\) in \(X\) are endpoints of a unique metric segment \([x, y] \in F\) \(([x, y]\) is an isometric image of the real line interval \([0, d(x, y)]\)). We shall denote by \(\beta x \oplus (1 - \beta)y\) the unique point \(z\) of \([x, y]\) which satisfies

\[
d(x, z) = (1 - \beta)d(x, y), \quad \text{and} \quad d(z, y) = \beta d(x, y),
\]

where \(\beta \in [0, 1]\). Such metric spaces with a family \(F\) of metric segments are usually called \textit{convex metric spaces} [28].
If we have

\[ d(\alpha p \oplus (1 - \alpha)x, \alpha q \oplus (1 - \alpha)y) \leq \alpha d(p, q) + (1 - \alpha)d(x, y), \]

for all \( p, q, x, y \) in \( X \), and \( \alpha \in [0, 1] \), then \( X \) is said to be a hyperbolic metric space (see [32]).
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Examples of hyperbolic metric space are:

- Normed linear spaces.
- Hilbert open unit ball equipped with the hyperbolic metric and the CAT(0) spaces.
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**Definition**

Let $(M, d)$ be a hyperbolic metric space. We say that $M$ is **uniformly convex** (in short, UC) if for any $a \in M$, for every $r > 0$, and for each $\epsilon > 0$

$$
\delta(r, \epsilon) = \inf \left\{ 1 - \frac{1}{r} d \left( \frac{1}{2} x \oplus \frac{1}{2} y, a \right); d(x, a) \leq r, d(y, a) \leq r, d(x, y) \geq r\epsilon \right\} > 0.
$$
Among the nice properties satisfied by uniformly convex hyperbolic metric space \((X, d)\) is the property \((R)\) [19] which says that if \(\{C_n\}\) is a decreasing sequence of nonempty bounded convex closed subsets of \(X\), then \(\bigcap_{n \geq 1} C_n \neq \emptyset\).
Lemma

Let $C$ be a nonempty closed convex subset of uniformly convex hyperbolic metric space $(X, d)$. Let $\tau : C \to [0, +\infty)$ be a type function, i.e., there exists a bounded sequence $\{x_n\} \in X$ such that

$$\tau(x) = \limsup_{n \to +\infty} d(x_n, x),$$

for any $x \in C$. Then $\tau$ is continuous. Since $X$ is hyperbolic, then $\tau$ is convex, i.e., the subset $\{x \in C; \tau(x) \leq r\}$ is convex for any $r \geq 0$. Moreover, there exists a unique minimum point $z \in C$ such that

$$\tau(z) = \inf\{\tau(x); x \in C\}.$$
Since monotone mappings do not in general have good global behavior, it is hard to expect some general fixed point theorems, like Kirk’s fixed point theorem, to extend easily. Therefore our approach was based on techniques that use constructive proofs.
Let \((X, d)\) be a hyperbolic metric space endowed with a partial order \(\preceq\). Throughout, we will assume that order intervals are convex and closed. Let \(C\) be a nonempty convex subset of \(X\) not reduced to one point. Let \(T : C \to C\) be monotone nonexpansive mapping. Fix \(\lambda \in (0, 1)\) and \(x_0 \in C\). Consider the \textit{Krasnoselskii-Ishikawa} [17, 25] iteration sequence \(\{x_n\}\) in \(C\) defined by

\[
x_{n+1} = (1 - \lambda) x_n \oplus \lambda T(x_n), \quad n \geq 0.
\]

(KIS)
Assume that $x_0$ and $T(x_0)$ are comparable, i.e. $x_0 \preceq T(x_0)$. Since order intervals are convex, we have $x_1 \preceq x_2 \preceq T(x_1)$. Since $T$ is monotone, we get $T(x_1) \preceq T(x_2)$. By induction, we will prove that

$$x_n \preceq x_{n+1} \preceq T(x_n) \preceq T(x_{n+1}),$$

for any $n \geq 1$, which implies, since $T$ is monotone nonexpansive,

$$d(T(x_{n+1}), T(x_n)) \leq d(x_{n+1}, x_n).$$
Lemma

Let \((X, d, \leq)\) be a POHMS having the above properties. Let \(C\) be a convex and bounded subset of \(X\) not reduced to one point. Let \(T : C \to C\) be a monotone nonexpansive mapping. Fix \(\lambda \in (0, 1)\) and \(x_0 \in C\) such that \(x_0\) and \(T(x_0)\) are comparable. Consider the sequence \(\{x_n\}\) in \(C\) defined by (KIS). Hence

\[
(GK) \quad (1 + n\lambda) \ d(T(x_i), x_i) \leq d(T(x_{i+n}), x_i) \\
+ (1 - \lambda)^{-n} \left( d(T(x_i), x_i) - d(T(x_{i+n}), x_{i+n}) \right),
\]

for any \(i, n \in \mathbb{N}\). Then we have

\[
\lim_{n \to +\infty} d(x_n, T(x_n)) = 0,
\]
Theorem

Let \((X, d, \preceq)\) be a partially ordered hyperbolic metric space as described above. Assume \((X, d)\) is uniformly convex. Let \(C\) be a nonempty convex closed bounded subset of \(X\) not reduced to one point. Let \(T : C \to C\) be a monotone nonexpansive mapping. Assume there exists \(x_0 \in C\) such that \(x_0\) and \(T(x_0)\) are comparable. Then \(T\) has a fixed point.


J. J. Nieto, R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to...*


