Approximate Global Minimum of the Difference of Increasing and Positively Homogeneous Functions

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Abstract

In this talk, we first obtain a formula for conjugate and $\varepsilon$-subdifferential of the difference of two abstract functions under a mild condition. Also, we characterize global $\varepsilon$-minimum of the difference of two abstract convex functions. Next, by using abstract Rockafellar’s antiderivative, we present the $\varepsilon$-subdifferential of abstract convex functions in terms of their subdifferential. Finally, as an application, we give a necessary and sufficient condition for (global) $\varepsilon$-minimum of the difference of two increasing and positively homogeneous (IPH) functions.
$X$ is a set.

\[ L := \{ \ell : X \to \mathbb{R} : \ell \text{ is a function} \} \]

is a set of real valued functions defined on $X$. We assume that $L$ contains the function $0_X$.

For each $\ell \in L$ and $c \in \mathbb{R}$, 

\[ h_{\ell,c}(x) := \ell(x) - c, \quad (x \in X). \]

The function $h_{\ell,c}$ is called $L$-affine.

$H_L$ : the set of all $L$-affine functions.
A function $f : X \rightarrow (-\infty, +\infty]$ is called $H$-convex ($H = L$, or $H = H_L$) if

$$f(x) = \sup\{h(x) : h \in \text{supp} (f, H)\}, \quad \forall x \in X,$$

where

$$\text{supp} (f, H) := \{h \in H : h \leq f\}$$

is called the support set of the function $f$. 
Let $U \subseteq H$ be a set of functions. A function $f \in U$ is called a maximal element of the set $U$, if $\tilde{f} \in U$ is such that $\tilde{f}(x) \geq f(x)$ for all $x \in X$, then, $\tilde{f}(x) = f(x)$ for all $x \in X$. 
For a function $f : X \to (-\infty, +\infty]$, define the Fenchel-Moreau $L$-conjugate $f^*_L$ of $f$ by

$$f^*_L(\ell) := \sup_{x \in X} (\ell(x) - f(x)), \quad \ell \in L.$$ 

The function $f^{**}_{L,X} := (f^*_L)^*_{X}$ is called the second conjugate (or biconjugate) of $f$ and, by the definition, we have

$$f^{**}_{L,X}(x) := \sup_{\ell \in L} (\ell(x) - f^*(\ell)), \quad x \in X.$$
Let $f : X \to (-\infty, +\infty]$ be a function and $x_0 \in \text{dom } f$. The $L$-subdifferential of $f$ is the set valued mapping $\partial_L f : X \rightrightarrows L$ is defined by

$$\partial_L f(x_0) := \{ \ell \in L : \ell(x) - \ell(x_0) \leq f(x) - f(x_0), \forall x \in X \}.$$ 

Given $\varepsilon \geq 0$, the $L$-$\varepsilon$-subdifferential of $f$ is the set valued mapping $\partial_{L, \varepsilon} f : X \rightrightarrows L$ is defined by

$$\partial_{L, \varepsilon} f(x_0) := \{ \ell \in L : \ell(x) - \ell(x_0) - \varepsilon \leq f(x) - f(x_0), \forall x \in X \}.$$ 

Also, for $x_0 \notin \text{dom } f$, we define $\partial_L f(x_0) = \partial_{L, \varepsilon} f(x_0) := \emptyset$. 
We can characterize the $L$-subdifferential of $f$ and the $L$-$\varepsilon$-subdifferential of $f$, as follows

$$\partial_L f(x_0) = \{ \ell \in L : f(x_0) + f^*_L(\ell) = \ell(x_0) \},$$

and

$$\partial_{L,\varepsilon} f(x_0) = \{ \ell \in L : f(x_0) + f^*_L(\ell) \leq \ell(x_0) + \varepsilon \}.$$

Also,

$$\partial_L f(x_0) = \bigcap_{\varepsilon \geq 0} \partial_{L,\varepsilon} f(x_0), \quad (x_0 \in \text{dom } f).$$
Definition: Let \( \varepsilon \geq 0 \) be given. A point \( x_0 \in X \) is said to be a (global) \( \varepsilon \)-minimum of the proper function \( f : X \to \mathbb{R}_{+\infty} \), if

\[
f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon.
\]
Now, we present a formula for conjugate of the difference of two abstract functions.

**Proposition:** Let $f, g : X \to \mathbb{R}_{+\infty}$ be proper functions such that $Dom(\partial_L g) = X$. Then,

$$ (f - g)_L^*(\ell) = \sup_{w \in dom(g_L^*)} (f_L^*(\ell + w) - g_L^*(w)), \forall \ell \in L. $$
Let $U$ and $W$ be two subsets of $L$. We use the following notation.

$$U \ominus W := \{ \ell \in L : \ell + W \subseteq U \}.$$ 

Theorem: Let $f, g : X \rightarrow \mathbb{R}_{+\infty}$ be functions such that $\text{Dom}(\partial_L g) = X$. Let $x_0 \in \text{dom} \ f$ and $\varepsilon \geq 0$. Then,

$$\partial_{L,\varepsilon}(f - g)(x_0) = \bigcap_{\delta \geq 0} \{ \partial_{L,\varepsilon + \delta} f(x_0) \ominus \partial_{L,\delta} g(x_0) \}.$$
Theorem: Let $f, g : X \to \mathbb{R}_{++}^\infty$ be $H_L$-convex functions such that $\text{Dom}(\partial L g) = X$. Then, $x_0$ is a (global) $\varepsilon$-minimum of $f - g$ if and only if

$$\partial_{L,\delta} g(x_0) \subseteq \partial_{L,\varepsilon + \delta} f(x_0), \ \forall \ \delta \geq 0.$$
Let $f : X \rightarrow \mathbb{R}_{+\infty}$ be a function. Let $\varepsilon \geq 0$ and $x_0 \in X$ be given. We consider a set $S_{\varepsilon}f(x_0)$ as follows

$$S_{\varepsilon}f(x_0) := \{ h \in \text{supp} (f; H_L) : h(x_0) = f(x_0) - \varepsilon \}.$$ 

In the following, we present the relation between $\partial_{L,\varepsilon}f(x_0)$ and $S_{\varepsilon}f(x_0)$. 
Proposition: Let \( f : X \rightarrow \mathbb{R}_{+\infty} \) be an \( H_L \)-convex function. Let \( x_0 \in X \) and \( \varepsilon \geq 0 \). Then, the following assertions are true.

(1) \( \ell \in \partial_{L,\varepsilon} f(x_0) \) if and only if \( h = (\ell, \ell(x_0) - f(x_0) + \varepsilon) \in S_{\varepsilon} f(x_0) \).

(2) An element \( \ell \in L \) is a maximal element of \( \partial_{L,\varepsilon} f(x_0) \) if and only if \( h = (\ell, \ell(x_0) - f(x_0) + \varepsilon) \) is a maximal element of \( S_{\varepsilon} f(x_0) \).
In the following, we give a necessary and sufficient condition for (global) \( \varepsilon \)-minimum of \( f - g \) in terms of elements of \( S_\varepsilon f(x_0) \) and \( S_\varepsilon g(x_0) \).

Proposition: Let \( f, g : X \to \mathbb{R}_+^\infty \) be \( H_L \)-convex functions such that \( \text{Dom}(\partial_L g) = X \). Let \( \varepsilon \geq 0 \) and \( x_0 \in X \). Then, \( x_0 \) is a (global) \( \varepsilon \)-minimum of \( f - g \) if and only if, for each \( \delta \geq 0 \), \( (\ell, \ell(x_0) - f(x_0) + \varepsilon + \delta) \in S_{\varepsilon + \delta} f(x_0) \) whenever \( (\ell, \ell(x_0) - g(x_0) + \delta) \in S_{\delta} g(x_0) \).
Now, we give the following condition:

**Condition (B):** Let \( H = L \), or \( H = H_L \). Let \( h(x) := \lim_{\alpha \in \Delta} h_\alpha(x) \) (\( x \in X \)), where \( \Delta \) is a directed set and \((h_\alpha)_{\alpha \in \Delta}\) is a net which is bounded from below. Then, either \( h \in H \), or \( h \equiv +\infty \).

**Theorem:** Suppose that \( H_L \) enjoys the condition (B). Let \( f : X \rightarrow \mathbb{R}_{+\infty} \) be an \( H_L \)-convex function. Let \( x_0 \in X \) and \( \varepsilon \geq 0 \). Then, for each element \( h := (\ell, \ell(x_0) - f(x_0) + \varepsilon) \in \mathcal{S}_{\varepsilon}f(x_0) \), there exists a maximal element \( \tilde{h} = (\ell, \ell - f(x_0) + \varepsilon) \in \mathcal{S}_{\varepsilon}f(x_0) \) such that \( \ell \leq \tilde{\ell} \) on \( X \) and \( \ell(x_0) = \tilde{\ell}(x_0) \).
In the following, we characterize (global) $\varepsilon$-minimum of $f - g$ in terms of maximal elements of $S_\varepsilon f(x_0)$ and $S_\varepsilon g(x_0)$.

Lemma: Suppose that $H_L$ enjoys the condition $(B)$. Let $f, g : X \rightarrow \mathbb{R}^{+\infty}$ be $H_L$-convex functions such that $\text{Dom}(\partial Lg) = X$. Let $\varepsilon \geq 0$ and $x_0 \in X$. Then, $x_0$ is a (global) $\varepsilon$-minimum of $f - g$ if and only if, for each $\delta \geq 0$ and each maximal element $(\ell, \ell(x_0) - g(x_0) + \delta) \in S_\delta g(x_0)$, there exists a maximal element $(\ell', \ell'(x_0) - f(x_0) + \varepsilon + \delta) \in S_{\varepsilon+\delta} f(x_0)$ such that $\ell \leq \ell'$ on $X$ and $\ell(x_0) = \ell'(x_0)$. 

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**$L$-$\varepsilon$-Subdifferential in terms of $L$-Subdifferential**

**Definition:** Let $M : X \rightharpoonup L$ be a set valued mapping and let $n \in \mathbb{N}$. We say that $M$ is $n$-$L$-monotone, if for any set of $n$ pairs $\{(x_i, \ell_i)\}_{i=1}^{n} \subseteq G(M)$ with $x_{n+1} := x_1$, we have

$$\sum_{i=1}^{n} [\ell_i(x_i) - \ell_i(x_{i+1})] \geq 0.$$ 

A mapping $M$ is said to be $L$-cyclically monotone, if it is $n$-$L$-monotone for all $n \in \mathbb{N}$. A 2-$L$-monotone mapping is simply called $L$-monotone. The mapping $M$ is said to be maximal $n$-$L$-monotone, if $G(M)$ has no proper $n$-$L$-cyclically monotone extension in $X \times L$. 

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Lemma: Let \( f : X \rightarrow \mathbb{R}_{+\infty} \) be a proper function. Then, \( \partial_L f \) is \( L \)-cyclically monotone.

Definition: Let \( M : X \rightrightarrows L \) be a set valued mapping. Let \( f : X \rightarrow \mathbb{R}_{+\infty} \) be a proper function. We say that \( f \) is an \( L \)-antiderivative of \( M \) whenever \( G(M) \subseteq G(\partial_L f) \).
Definition: Let $M : X \rightrightarrows L$ be a set valued mapping and $s \in \text{Dom}(M)$. Define $L$-Rockafellar function $R_{[M,s]} : X \rightarrow \mathbb{R}_+ \cup \{\infty\}$ associated with $M$ by

$$R_{[M,s]}(x) := \sup_{\{(x_i, \ell_i)\}^n_{i=1} \subseteq G(M)} \sum_{i=1}^n [\ell_i(x_{i+1}) - \ell_i(x_i)].$$

$x_1 = s, x_{n+1} := x$
Theorem: Let $f : X \rightarrow \mathbb{R}_+\infty$ be a proper $H_L$-convex function such that $\text{Dom}(\partial_L f) = \text{dom} f$. Suppose that $\partial_L f$ admits a unique (up to an additive constant) $L$-antiderivative. Let $x \in \text{dom} f$ and $\varepsilon \geq 0$. Then,

$$\partial_L \varepsilon f(x) = \left\{ \ell \in L : \ell(x) - \ell(x_0) + \ell_m(x_m) - \ell_m(x) + \sum_{i=0}^{m-1} [\ell_i(x_i) - \ell_i(x_{i+1})] \geq -\varepsilon, \forall (x_i, \ell_i) \in G(\partial_L f), \ (i = 0, ..., m) \right\}.$$

In the following, by using the above theorem, we give a characterization of (global) $\varepsilon$-minimum of $f - g$, where $f$ and $g$ are $H_L$-convex functions.
Theorem: Let \( f, g : X \rightarrow \mathbb{R}_{+\infty} \) be \( H_L \)-convex functions such that \( \text{Dom}(\partial_L f) = \text{dom} f \) and \( \partial_L f \) admits a unique (up to an additive constant) \( L \)-antiderivative. Let \( y_0 \in \text{Dom}(\partial_L g) \) and \( \ell_0 \in \partial_L g(y_0) \). Then, \( y_0 \) is a (global) \( \varepsilon \)-minimum of \( f - g \) if and only if

\[
\sum_{i=0}^{r}[\ell_i(x_i) - \ell_i(x_{i+1})] + \sum_{j=0}^{s}[\ell'_j(y_j) - \ell'_j(y_{j+1})] + \ell_r(x_0) - \ell_r(y_0) + \ell'_s(y_0) - \ell'_s(x_0) \geq -\varepsilon, \quad \forall \ (x_i, \ell_i) \in G(\partial_L f), \ (i = 0, \ldots, r), \ x_{r+1} := x_0, \\
\forall \ (y_j, \ell'_j) \in G(\partial_L g), \ (j = 0, \ldots, s), \ y_{s+1} := y_0.
\]
$\varepsilon$-Minimum of the Difference of Increasing and Positively Homogeneous Functions

Let $X$ be a real topological vector space. We assume that $X$ is equipped with a closed convex pointed cone $S \subseteq X$. We say $x \leq y$ if and only if $y - x \in S$.

A function $p : X \to [-\infty, +\infty]$ is IPH if $p$ is an increasing and positively homogeneous function.
Now, consider the function $l : X \times X \to [0, +\infty]$ is defined by

$$l(x, y) := \max\{\lambda \geq 0 : \lambda y \leq x\}, \ \forall \ x, y \in X.$$ 

Define $L := \{l_y : y \in X \setminus (-S)\} \cup \{0\}$, where $l_y(x) := l(x, y)$ for all $x \in X$ and all $y \in X$. Note that $l_y$ is an IPH function for each $y \in X$, and every non-negative IPH function is $L$-convex.
We have
\[
\text{supp} \ (p, L) = \{ l_y \in L : p(y) \geq 1 \}.
\]

Let \( x_0 \in X \) and \( p(x_0) \neq 0, +\infty \). We can characterize \( \partial_{LP}(x_0) \) as follows,
\[
\partial_{LP}(x_0) = \{ l_y \in L : p(x_0) + p^*_L(l_y) = l_y(x_0) \}
\]
\[
= \{ l_y \in \text{supp} \ (p, L) : p(x_0) = l_y(x_0) \}.
\]
Moreover, if \( x_0 \in dom \ p \) and \( \varepsilon \geq 0 \), we have

\[
\partial_{L,\varepsilon} p(x_0) = \{ l_y \in L : p(x_0) + p_L^*(l_y) \leq l_y(x_0) + \varepsilon \}
\]

\[
= \{ l_y \in supp (p, L) : p(x_0) \leq l_y(x_0) + \varepsilon \}.
\]
Remark: Let $p : X \rightarrow [0, +\infty)$ be an IPH function. Then, $\text{Dom}(\partial Lp) = X$. Indeed, Assume that $p(x) \neq 0$ (note that in this case, $p(x) > 0$, and hence, $x \notin -S$). Thus, one has $l \frac{x}{p(x)} \in \partial Lp(x)$. If $p(x) = 0$, then, $0 \in \partial Lp(x)$. Also, since $p$ is $L$-convex, it follows that $p$ is $H_L$-convex, so, $\partial_{L, \varepsilon} p(x_0) \neq \emptyset$ for all $x_0 \in \text{dom}(p)$ and all $\varepsilon \geq 0$. 
Theorem: Let \( p : X \rightarrow [0, +\infty] \) be an IPH function. Then, \( \partial_{LP} \) is a maximal \( L \)-monotone operator.

Proposition: Let \( p : X \rightarrow [0, +\infty] \) be an IPH function. Then, \( \partial_{LP} \) admits a unique (up to an additive constant) IPH \( L \)-antiderivative.
Theorem: Let $p, q \rightarrow [0, +\infty)$ be IPH functions. Let $\epsilon \geq 0$ and $x \in X$. Consider the following systems of inequalities:

\[
\begin{align*}
Q_q : \begin{cases}
  l_y(x) = q(x), \\
  q(y) = 1, \ y \in X.
\end{cases} \\
Q_p : \begin{cases}
  p(x) \leq l_y(x) + \epsilon, \\
  p(y) \geq 1, \ y \in X.
\end{cases}
\end{align*}
\]

Then, $x$ is a (global) $\epsilon$-minimum of $p - q$ if and only if the solutions set of $(Q_p)$ contains the solutions set of $(Q_q)$. 

Theorem: Let $p, q : X \rightarrow [0, +\infty)$ be IPH functions. Let $\varepsilon \geq 0$ and $y \in X$. Then, $y$ is a (global) $\varepsilon$-minimum of $p - q$ if and only if

$$p(x) + q(y) - (l_z(y) + l_t(x)) \geq -\varepsilon, \ \forall \ l_z \in \partial_{Lp}(x), \ \forall \ l_t \in \partial_{Lq}(y), \ (x \in X).$$
References


